Fermi National Accelerator Laboratory

FERMILAB–CONF–88/130–T
September, 1988

On the Algebraic Structure of Integrable Systems in Multidimensions *

Jean Michel Maillet†
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, Illinois, 60510, USA

Frank Nijhoff
Department of Mathematics and Computer Science
Clarkson University, Potsdam, New York, 13676, USA

Abstract

The d-simplex equations are multidimensional generalizations of the Yang-Baxter (triangle) equations. We discuss the algebraic structure of the hierarchy of the d-simplex equations and their relevance to the notion of quantum and classical integrability. A multidimensional generalization of the notion of lax pair is given.

1. The development of a genuinely multidimensional notion of integrability is one of the major outstanding problems in the theory of integrable systems.In this note we wish to report on the developments of a program of study of Integrable Systems in dimensions higher than two that we initiated in [1].One of the main idea, from our point of view, is that the notion of integrability is intimately linked to the question of the possibility of constructing an overdetermined, but solvable, system of equations in a given dimension which does not reduce trivially to a lower dimensional one. On the other hand, it is well known that the Yang-Baxter (triangle) equations are the fundamental building blocks in the theory of two-dimensional Integrable Systems[2].

*To appear in the proceedings of the XVII ICGTMP, 1988, St. Adele, Canada.

†Address after October 1, 1988: CERN, Division Theorique, CH-1211 Geneve 23, Switzerland

Therefore, it is natural, if one is interested in extending the algebraic structure of Integrable non-linear Systems to dimensions higher than two to consider the possibility of generalizing the Yang-Baxter equations. Such a generalization exists. It was proposed by Zamolodchikov in 1980 for the case $d = 3$[3] and extended to arbitrary d in ref. [4]. From their very nice geometrical nature, these equations are called the d-simplex equations.

The idea we wish to put forward in this note is that it is useful to consider the complete hierarchy of the d-simplex equations in order to get an insight into the algebraic structure of its individual members. In particular, we will show how this hierarchy develops and how this concept that underlies the logical construction of the hierarchy step by step is also useful when trying to generalize the notion of classical integrability and of linear system for higher dimensional Integrable Models. A more detailed description of our ideas and results may be found in refs. [5-6].

2. Our starting point will be a (very brief) description of the d-simplex equations. They can be written as:

$$S^{(1)} * S^{(2)} * \dots * S^{(d+1)} = S^{(d+1)} * \dots * S^{(2)} * S^{(1)} \tag{1}$$

where each object $S^{(k)}$ is a tensor acting as a linear operator in the tensor product of d vector spaces, i.e., $S^{(k)} \equiv S^{k_1 \cdots k_d}_{k'_1 \cdots k'_d}$. Now, the product in eq.(1) is such that every $S^{(k)}$ is connected, via a usual matrix product, to each of the other $S^{(j)}, j \neq k$, once and only once. So, eq.(1) expresses that such a product of $(d+1)$ operators is the same when reversing at the same time all these products. Hence, eq.(1) may be seen as a generalization of matrix commutativity for multiple indices objects[5,6]. In addition, each object$S^{(k)}$ is depending on d continuum parameters, $\frac{d(d+1)}{2} - 1$ being independant in eq.(1), (see refs. [5,6] for more details). Let us give some examples of eq. (1) for $d = 1, 2, 3 (S^{(1)} \equiv A, S^{(2)} \equiv B$, etc.):

$$\mathbf{d = 1} : A^k_i . B^j_k = B^k_i . A^j_k \qquad \text{(commutativity condition)}$$

$$\mathbf{d = 2} : A^{k_1 k_2}_{i_1 i_2} . B^{j_1 k_3}_{k_1 i_3} . C^{j_2 j_3}_{k_2 k_3} = C^{k_2 k_3}_{i_2 i_3} . B^{k_1 j_3}_{i_1 k_3} . A^{j_1 j_2}_{k_1 k_2} \qquad \text{(Yang − Baxter equation)}$$

$$\mathbf{d = 3} : A^{k_1 k_2 k_3}_{i_1 i_2 i_3} . B^{j_1 k_4 k_5}_{k_1 i_4 i_5} . C^{j_2 j_4 k_6}_{k_2 k_4 i_6} . D^{j_3 j_5 j_6}_{k_3 k_5 k_6} = D^{k_3 k_5 k_6}_{i_3 i_5 i_6} . C^{k_2 k_4 j_6}_{i_2 i_4 k_6} . B^{k_1 j_4 j_5}_{i_1 k_4 k_5} . A^{j_1 j_2 j_3}_{k_1 k_2 k_3}$$

(Zamolodchikov equation)

Our main purpose now is to describe the principle that enable us to generate the $(d+1)$–simplex equation from the d–simplex equation. Let us consider $(d+1)$ collections of objects ${}^{\alpha_k}S^{(k)}$ that can be coupled as in the r.h.s. of eq.(1), but that have an additional index ${}^{\alpha_k}$. We will now suppose that these tensors do not verify the d–simplex equation (1) but rather that there exists a tensor, say $R^{\beta_1\cdots\beta_{d+1}}_{\alpha_1\cdots\alpha_{d+1}}$ that describes the breaking of eq. (1) for the objects ${}^{\alpha_k}S^{(k)}$ in the following way:

$$R^{\beta_1\cdots\beta_{d+1}}_{\alpha_1\cdots\alpha_{d+1}}\cdot{}^{\alpha_1}S^{(1)}*\ldots*{}^{\alpha_{d+1}}S^{(d+1)}={}^{\beta_{d+1}}S^{(d+1)}*\ldots*{}^{\beta_1}S^{(1)} \tag{2}$$

where the sum over $\alpha_i, i=1,\cdots,d+1$ is understood. If $R^{\beta_1\cdots\beta_{d+1}}_{\alpha_1\cdots\alpha_{d+1}}\equiv\delta^{\beta_1}_{\alpha_1}\cdots\delta^{\beta_{d+1}}_{\alpha_{d+1}}$ (identity tensor), eq. (2) would reduce to eq. (1). However, in general, if R is not the identity tensor, it describes a non–trivial breaking of the relation (1). Then it is possible to show that, requiring the associativity of operator product in eq. (2), the tensor $R^{\beta_1\cdots\beta_{d+1}}_{\alpha_1\cdots\alpha_{d+1}}$ has to satisfy the $(d+1)$–simplex equation [6]. The most simple example is given by the breaking of the 1–simplex (commutativity) equation that leads to the Yang–Baxter (2–simplex) equation as compatibility condition. From another point of view, eq. (2) leads to the generalization of the braid group for higher dimensional objects[6].

3. The concept of "breaking" we have just developed is, in fact, very suggestive, and it is possible along this line of thinking to generalize the notion of Lax equations for higher dimensional classical Integrable Systems. For simplicity we will only consider lattice models here, and restrict ourselves to the case $d=3$ even though our construction works recursively for any d. In two dimensions the Lax system reads:

$$\phi^{\ell}(m,n)=L_{\ell}(m,n)\phi(m,n) \tag{3}$$

where $\ell=1,2,$ L_ℓ is a matrix depending on the fields and on a spectral parameter, and $\phi(m,n)$ is a solution of the Lax linear system (3) at a point (m,n) on the two–dimensional lattice. The upperscript will denote a shift in direction ℓ, namely $\phi^1(m,n)\equiv\phi_{(m+1,n)}$ and $\phi^2_{(m,n)}\equiv\phi_{(m,n+1)}$. In the following the site index (m,n) will be omitted. The compatibility condition for eq. (3) is given by:

$$L^{\ell'}_{\ell}\cdot L_{\ell'}=L^{\ell}_{\ell'}\cdot L_{\ell} \tag{4}$$

The "breaking" principle we developed in section 2 tells us now that what to do to construct the $d=3$ generalization of eqs. (3,4). We define new objects $L_\ell, \ell=1,2,3$

that are tensors, each acting on the tensor product of two spaces rather than being usual matrices as before. We now suppose that eq. (4) is no longer satisfied for such objects, but that there exists a new tensor $\chi_{\ell\ell'}$ that describes the breaking of this relation:

$$(\chi_{\ell\ell'})^{ij}_{kh} \left(L^{\ell'}_{\ell}\right)^{ka}_{i'b} (L_{\ell'})^{hb}_{j'c} = \left(L^{\ell}_{\ell'}\right)^{ja}_{j'b} (L_{\ell})^{ib}_{i'c} \tag{5}$$

This is our proposal for a $d = 3$ generalization of the Lax equations on a lattice[6]. Then using the * product of eq. (1) it is possible to write the compatibility condition associated to eq. (5):

$$\chi_{\ell\ell'} * \chi^{\ell'}_{\ell\ell''} * \chi_{\ell'\ell''} = \chi^{\ell}_{\ell'\ell''} * \chi_{\ell\ell''} * \chi^{\ell''}_{\ell\ell'} \tag{6}$$

The products between the χ are the same as in the Yang–Baxter equation ($d = 2$), but, now, because the χ are depending on fields defined on a three–dimensional lattice, the shifts (upper indices) are non trivial. Indeed, eq. (6) realizes a gauging of the usual Yang–Baxter equation, and hence of the quantum group associated to it[7].

4. To conclude, we wish to emphasize that our central generating principle for both aspects of multidimensional integrability we have briefly treated here, consists in the concept of "breaking." It is useful to note that in particular, this new concept provides us with a systemmatic way of constructing the d–simplex equations hierarchy together with generalization of the two–dimensional Lax equations in any dimensions.

References

[1] F.W. Nijhoff and J.M. Maillet, in Proc. of IV Int. Workshop on Non Linear Evolution Equations, ed. J. Leon, (World Scientific, Singapore, (1988).

[2] R.J. Baxter, "Exactly Solved Models in Statistical Mechanics," (Academic Press, London, 1982); L.D. Faddeev, in Les Houches Lectures, 1982, eds. R. Stora and J.B. Zuber, (North Holland, 1983).

[3] A.B. Zamolodchikov, Comm. Math. Phys. 79, 489 (1981).

[4] V.V. Bazhanov and Yu. G. Stroganov, Nucl. Phys. B230 [FS10], 435 (1984).

[5] J.M. Maillet and F.W. Nijhoff, "The Tetrahedron Equation and the Four–Simplex Equation," Fermilab–Pub–88/71–T.

[6] J.M. Maillet and F.W. Nijhoff, to be published.

[7] J.M. Maillet and F.W. Nijhoff, "Gauging the Quantum Groups," to be published.